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A Hamiltonian approach to Lagrangian Noether transformations

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Abstract. It is proven that each Lagrangian Noether symmetry—rigid or gauge—can be easily obtained from a kind of 'Hamiltonian generator', which is a conserved quantity satisfying a simple condition. This yields a procedure to construct Lagrangian gauge transformations. It is also shown that some regularity conditions are needed in order to assure the existence of Hamiltonian gauge generators: we exhibit an example which has no such generators, though Noether gauge transformations can be constructed for it. We apply our method to obtain the covariant gauge transformations of the bosonic string from its Hamiltonian constraints.

1. Introduction

Equations of motion for constrained systems, in Hamiltonian or Lagrangian formulation, generally exhibit terms which depend linearly on arbitrary functions of time. Owing to this, there is a family of dynamic trajectories passing through every admissible initial condition. This family corresponds to a unique physical motion and its members become related by gauge transformations. Then it is convenient to find a complete set of independent gauge transformations in order to elucidate the gauge freedom of the theory, and therefore its physical contents, especially in regard to quantization and BRST formalism.

Here we shall consider Hamiltonian and Lagrangian constrained formalisms built up from a time independent singular Lagrangian§ $L(q, v)$ defined in the tangent (velocity) space TQ of a certain configuration space Q. Then the Hamiltonian formalism is introduced in phase space T^*Q using Dirac's method [1].

It will be assumed that the Hessian matrix $W = \partial^2 L/\partial v\, \partial v$ has 'constant rank'. Under this hypothesis it is possible to show (local) equivalence between the Lagrangian and Hamiltonian formalisms, in the sense that they lead to the same dynamic trajectories in configuration space [2].

However this equivalence has a non-trivial content when other aspects of the formalism are taken into account. This is the case, for instance, of constraints, degrees of freedom, presence of arbitrary functions in the dynamics, rigid symmetries

§ Indices of coordinates will be always omitted.

and gauge transformations. The reason is that for singular Lagrangians the Legendre transformation $FL: \mathrm{T}Q \to \mathrm{T}^*Q$ defined by

$$FL(q, v) = (q, \hat{p}) \qquad \hat{p} = \frac{\partial L}{\partial v} \tag{1.1}$$

cannot be inverted.

In particular, let us consider the case of gauge transformations. Once the respective constraint surfaces have been determined—through Dirac's stabilization algorithm—both Lagrangian and Hamiltonian equations of motion exhibit the same number of arbitrary functions of time. This number equals the number of (final) first class primary constraints in Hamiltonian formalism. Accordingly, the same number of independent gauge transformations is expected to describe completely the gauge freedom of the theory [3].

If this set is known in phase space, a simple pull-back operation through the Legendre transformation (i.e. substitution of the momenta by their Lagrangian expressions) leads to the correct number of independent Lagrangian gauge transformations. But the converse operation (to start with Lagrangian gauge transformations to obtain the Hamiltonian transformations) is not always possible because there are functions in velocity space which are not projectable to phase space (i.e. they cannot be expressed in terms of positions and momenta). This 'inequivalence' between the Lagrangian and Hamiltonian formalisms is again a clear consequence of the Legendre transformation not being invertible.

In the study of constrained systems, obtaining gauge transformations has always had a relevant place. Some recent results on this topic are collected in section 2, in particular the construction of Hamiltonian gauge generators. Section 2 contains also some notation and technical results.

The purpose of this paper is twofold:

(1) We shall establish a general result on—not necessarily projectable—Lagrangian Noether symmetries, including either rigid or gauge symmetries: they can be always derived from a kind of 'Hamiltonian generator' which fulfils a certain condition. The problem of searching Noether transformations is thus converted into a simpler one. This is done in section 3. The main tool is an unambiguous evolution operator, some of whose properties are listed in section 2.

(2) As explained in section 2, the existence of a complete set of independent Hamiltonian gauge generators can be proven under several regularity assumptions. Some of them cannot be removed: in section 4 we present an example whose Hamiltonian constraints are first class, which has no Hamiltonian gauge generators but allows for a non-projectable Lagrangian Noether gauge transformation. This is found using the method introduced in section 3.

We also apply our method to find the covariant gauge transformations of the bosonic string from its Hamiltonian constraints. The last section is devoted to comments and conclusions.

2. Preliminary results

2.1. Projectability, constraints and evolution operators

Let us first recall two definitions. Given a function $g(q, p)$ in phase space, its pull-back (through the Legendre transformation FL) is the function $FL^*(g)$ in velocity space obtained by substituting the momenta by their Lagrangian expression

$$\mathrm{F}L^*(g)(q,v) = g\left(q, \frac{\partial L}{\partial v}\right). \tag{2.1}$$

A function $f(q,v)$ in velocity space is called (FL-)projectable if it can be obtained as the pull-back of a certain function $g(q,p)$ in phase space.

The following result [4] will be needed in the next section: if γ_μ ($\mu = 1,\ldots,m_1$) are a basis for the null vectors of the Hessian matrix W, then the necessary and sufficient condition for a function $f(q,v)$ in TQ to be projectable to T$^*Q$ is

$$\Gamma_\mu \cdot f = 0 \quad (\mu = 1,\ldots,m_1) \tag{2.2}$$

where $\Gamma_\mu := \gamma_\mu \partial/\partial v$. The basis γ_μ can be taken as [2]

$$\gamma_\mu := \mathrm{F}L^*\left(\frac{\partial \phi_\mu^0}{\partial p}\right) \tag{2.3}$$

where ϕ_μ^0 ($\mu = 1,\ldots,m_1$) are a complete set of independent primary Hamiltonian constraints.

The time-derivative operator in Lagrangian formalism is

$$\frac{\mathrm{d}}{\mathrm{d}t} = v\frac{\partial}{\partial q} + a\frac{\partial}{\partial v} + \frac{\partial}{\partial t} \tag{2.4}$$

with the acceleration a as an independent variable. Then the Euler-Lagrange equations can be written $[L]_{(q,\dot q,\ddot q)} = 0$, where we have defined

$$[L] := \frac{\partial L}{\partial q} - \frac{\mathrm{d}\hat p}{\mathrm{d}t} = \alpha - aW \tag{2.5}$$

with $\alpha = \dfrac{\partial L}{\partial q} - v\dfrac{\partial^2 L}{\partial q\,\partial v}$. The primary Lagrangian constraints arising from it are

$$\chi_\mu^1 := \alpha \cdot \gamma_\mu. \tag{2.6}$$

It is convenient in what follows to consider an evolution operator K which takes a function $g(q,p;t)$ in phase space and gives its time-derivative $(K\cdot g)(q,v;t)$ in velocity space

$$K\cdot g := v\cdot \mathrm{F}L^*\left(\frac{\partial g}{\partial q}\right) + \frac{\partial L}{\partial q}\cdot \mathrm{F}L^*\left(\frac{\partial g}{\partial p}\right) + \mathrm{F}L^*\left(\frac{\partial g}{\partial t}\right). \tag{2.7}$$

This operator was introduced in [2]. It establishes interesting connections between Hamiltonian and Lagrangian formalisms. For instance, all the Lagrangian constraints are obtained by applying it to the Hamiltonian constraints [5]; in particular, for the primary constraints,

$$\chi_\mu^1 = K\cdot \phi_\mu^0. \tag{2.8}$$

On the other hand the geometric formulation of K allows us to write the Euler-Lagrange equations in an intrinsic way [6].

The operator K can be given several interesting expressions. Now we shall use

$$K\cdot g = [L]\cdot \mathrm{F}L^*\left(\frac{\partial g}{\partial p}\right) + \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{F}L^*(g)) \tag{2.9}$$

whose proof is direct using the chain rule.

2.2. Gauge transformations

Here we quote some results from [3]. To be precise, we call dynamic symmetry transformations those transformations which map solutions of the equations of motion into solutions, either in Lagrangian or in Hamiltonian formalism.

The evolution operator K allows us to write, in a very compact form, the necessary and sufficient condition for a function $G_{\rm H}(q,p;t)$ to generate, through Poisson bracket

$$\delta f := \{f, G_{\rm H}\} \tag{2.10}$$

an infinitesimal dynamic symmetry transformation in Hamiltonian formalism. This condition is†

$$K \cdot G_{\rm H} \underset{V_f}{\cong} 0 \tag{2.11}$$

where V_f is the surface defined by all the Lagrangian constraints in velocity space. Equivalently, $G_{\rm H}$ is a first class function and satisfies

$$\{G_{\rm H}, H\} + \frac{\partial G_{\rm H}}{\partial t} \underset{M_f}{\cong} \text{PFC} \tag{2.12a}$$

$$\{\text{PFC}, G_{\rm H}\} \underset{M_f}{\cong} \text{PFC} \tag{2.12b}$$

where M_f is the surface defined by all the Hamiltonian constraints in phase space and PFC stands for any primary first class Hamiltonian constraint.

More particularly, we call gauge transformation a dynamic symmetry transformation which depends on arbitrary functions of time. The general form for the (infinitesimal) generators of Hamiltonian gauge transformations can be taken as

$$G_{\rm H}(q,p;t) = \sum_{k\geqslant 0} \epsilon^{(-k)}(t)\, G_k(q,p) \tag{2.13}$$

where $\epsilon^{(-k)}$ is a kth primitive of an arbitrary function of time ϵ.

Then, to find an effective gauge generator, the characterization (2.11) or (2.12) of $G_{\rm H}$ as a dynamic symmetry transformation splits yielding the following constructive algorithm, where strong equalities have been changed to normal equalities [3]:

$$G_0 = \text{PFC} \tag{2.14a}$$

$$\{G_k, H\} + G_{k+1} = \text{PFC} \tag{2.14b}$$

$$\{\text{PFC}, G_k\} \underset{M_f}{\cong} \text{PFC}. \tag{2.14c}$$

It is noticed, therefore, that though there may be second class constraints, the generators of Hamiltonian gauge transformations are built up of first class constraints, and, according to (2.14a), are 'headed' by a primary constraint.

† $f \underset{M}{\simeq} 0$ means $f = 0$ on M (Dirac's weak equality). $f \underset{M}{\cong} 0$ means $f \underset{M}{\simeq} 0$ and $\mathrm{d}f \underset{M}{\simeq} 0$ (Dirac's strong equality).

Other authors have considered similar generators of the form (2.13) when there are no second class constraints [7–9]; they have used the derivatives of ϵ rather than its primitives, and have arrived to a restricted form of (2.14). There is also an approach where the Lagrange's multipliers of the constraints are considered as dynamic variables [10]. A functional approach, more general, is also possible; see for instance [11, 12].

The question arises as to the existence of a basis of primary first class Hamiltonian constraints such that the preceding algorithm can be carried out. Concerning this, a theorem has been recently proven [13] which, under some additional regularity conditions (namely, the constancy of the rank of Poisson brackets among constraints and the non-appearance of ineffective constraints), establishes the existence of a complete set of independent Hamiltonian gauge generators of the form above. Its number, therefore, equals the number of (final) first class primary constraints. On the other hand, their pull-back constitutes a complete set of Lagrangian gauge transformations.

3. Noether's transformations

3.1. Construction of Noether's transformations

We consider an infinitesimal transformation $\delta q(q, v; t)$ in velocity space TQ, with a possible functional dependence on arbitrary functions of time. When δL is a total derivative (i.e., the action is invariant up to boundary terms), Noether's theorem guarantees that δq maps solutions into solutions. Such a δq is called a Noether transformation, and a conserved quantity arises from it [14–19]. Some recent papers dealing with geometric aspects of Noether's theorem are [20, 21]. So let us consider

$$\delta L = \frac{\mathrm{d}F}{\mathrm{d}t}. \tag{3.1}$$

This can be equivalently written as

$$[L] \cdot \delta q + \frac{\mathrm{d}G}{\mathrm{d}t} = 0 \tag{3.2}$$

with the conserved quantity $G = (\partial L/\partial v)\delta q - F$.

Since the acceleration a appears linearly in (3.2), its coefficient must vanish [22]

$$-W \cdot \delta q + \frac{\partial G}{\partial v} = 0. \tag{3.3}$$

By contraction with the γ_μ this implies $\Gamma_\mu \cdot G = 0$ for $\mu = 1, \ldots, m_1$. We conclude from (2.2) that G is a projectable function

$$G = FL^*(G_\mathrm{H}) \tag{3.4}$$

for a certain function $G_\mathrm{H}(q, p; t)$ in phase space, determined up to primary Hamiltonian constraints. Then the relation (3.3) also reads

$$W \cdot \left(\delta q - FL^* \left(\frac{\partial G_\mathrm{H}}{\partial p}\right)\right) = 0.$$

Therefore the term in parentheses is a null vector of W. So, using the basis (2.3), there exist uniquely defined functions $r^\mu(q,v;t)$ $(\mu = 1,\dots,m_1)$ such that

$$\delta q = \mathrm{F}L^* \left(\frac{\partial G_\mathrm{H}}{\partial p}\right) + r^\mu \gamma_\mu \tag{3.5}$$

which can also be written more symmetrically as

$$\delta q = \mathrm{F}L^*\{q, G_\mathrm{H}\} + r^\mu\, \mathrm{F}L^*\{q, \phi^0_\mu\}. \tag{3.6}$$

Introducing (3.5) into (3.2), we obtain

$$[L]\cdot\left(\mathrm{F}L^*\left(\frac{\partial G_\mathrm{H}}{\partial p}\right) + r^\mu\gamma_\mu\right) + \frac{\mathrm{dF}L^*(G_\mathrm{H})}{\mathrm{d}t} = 0$$

which, using expression (2.9), definition (2.5) and the definition of the primary Lagrangian constraints (2.6), becomes

$$K\cdot G_\mathrm{H} + r^\mu \chi^1_\mu = 0. \tag{3.7}$$

Then, if we define V_1 as the primary Lagrangian constraint surface, (3.7) can be written without any reference to the functions r^μ as

$$K\cdot G_\mathrm{H} \underset{V_1}{\simeq} 0. \tag{3.8}$$

Thus, starting from a Noether transformation (3.2), we have obtained a neat characterisation (3.8) of a function $G_\mathrm{H}(q,p;t)$ which carries all the information of the Noether transformation. Notice that addition of primary Hamiltonian constraints to G_H does not modify (3.8).

Conversely, we can reconstruct δq from (3.8), in order to satisfy (3.2), as follows. First the functions r^μ are determined in order to satisfy (3.7). Indeed, they are determined up to certain combinations of primary Lagrangian constraints; additional underdetermination appears when the primary Lagrangian constraints χ^1_μ are not independent. Then define δq through (3.6). The function G is also recovered as $G = \mathrm{F}L^*(G_\mathrm{H})$. With these definitions (3.2) is automaticaly fulfilled.

We have proven, therefore:

Theorem. Every Lagrangian Noether infinitesimal transformation $\delta q(q,v;t)$ can be obtained through (3.6) from a function $G_\mathrm{H}(q,p;t)$ satisfying $K\cdot G_\mathrm{H} \underset{V_1}{\simeq} 0$. Conversely, any function $G_\mathrm{H}(q,p;t)$ satisfying this relation generates in the same way a Lagrangian Noether transformation.

This G_H is the 'Hamiltonian generator in the sense of (3.6)' for δq which we were looking for. Notice, however, that such G_H is not necessarily a Hamiltonian gauge generator, since in general it does not satisfy (2.11).

For the sake of completeness we notice that the transformations of the momenta—as functions $\hat{p} = \partial L/\partial v$—can be shown to be

$$\delta\hat{p} = \mathrm{F}L^*\{p, G_\mathrm{H}\} + r^\mu\, \mathrm{F}L^*\{p,\phi^0_\mu\} - [L]\partial\delta q/\partial v \tag{3.9}$$

in full accordance with (3.6) when the equations of motion are taken into account.

3.2. Projectability of Noether's transformations

Now we consider a given function $G_{\rm H}$ satisfying (3.8). Two different cases can appear:

(1) The functions r^μ determined by (3.7) are projectable: $r^\mu = FL^*(r^\mu_{\rm H})$. Equivalently, δq is projectable. Then, if we redefine $\overline{G}_{\rm H} := G_{\rm H} + r^\mu_{\rm H}\phi^0_\mu$, equation (3.8) becomes

$$K \cdot \overline{G}_{\rm H} = 0.$$

This is the case studied in [23]. The infinitesimal transformation reads

$$\delta q = FL^*\{q, \overline{G}_{\rm H}\}.$$

Now $\overline{G}_{\rm H}$ is also a generator of infinitesimal dynamic symmetry transformations in Hamiltonian formalism, since it satisfies (2.11).

(2) The functions r^μ determined by (3.7) are not projectable. This is a new case not considered before. Now we have a Noether transformation which is not the pull-back of a Hamiltonian symmetry transformation. In the next section we present an example of this kind of transformation.

It is worth noticing that our characterization (3.7) can be weakened by substituting normal equalities for Dirac's strong equalities. Our results hold in this more general situation after changing normal equalities to strong equalities in equations as (3.6), (3.1) or (3.2).

3.3. Lagrangian gauge transformations

Finally, let us consider the particular case of Noether gauge transformations. If a 'generator' $G_{\rm H}$ of the form (2.13) is proposed, then condition (3.8) can be analysed to obtain a recursive algorithm in the same way that (2.11) leads to (2.14). We have

$$K \cdot G_{\rm H} = \dot{\epsilon} G^0_{\rm H} + \sum_{k \geqslant 0} \epsilon^{(-k)}(K \cdot G^k_{\rm H} + FL^*(G^{k+1}_{\rm H}))$$

so the arbitrariness of ϵ implies the recursive relations

$$FL^*(G^0_{\rm H}) \underset{V_1}{\simeq} 0 \tag{3.10a}$$

$$FL^*(G^{k+1}_{\rm H}) \underset{V_1}{\simeq} -K \cdot G^k_{\rm H}. \tag{3.10b}$$

This also shows that $G_{\rm H}$ is made up of Hamiltonian constraints. See also [24] for another construction of Lagrangian gauge transformations, in which the null vectors γ_μ play a role similar to that in this paper.

4. An example of a Lagrangian gauge transformation without a Hamiltonian counterpart

Here we discuss in detail a simple model which has no Hamiltonian gauge generators but exhibits, at the Lagrangian level, Noether gauge transformations†.

† A more physical example exhibiting similar features is provided by the second-order Lagrangian describing a relativistic particle with curvature.

We start with the singular Lagrangian

$$L = -\frac{1}{\alpha}\dot{\boldsymbol{y}} \cdot (\dot{\boldsymbol{x}} + \beta\dot{\boldsymbol{y}}). \tag{4.1}$$

Configuration space is described by the scalar coordinates α and β, assumed to take non-zero values, and the vector coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$, belonging to a vector space with an indefinite scalar product (Minkowski space, for instance).

The Hamiltonian analysis of this Lagrangian leads to the primary constraints

$$p_\alpha \simeq 0 \qquad p_\beta \simeq 0$$

which are first class. The Hamiltonian can be taken as

$$H = -\alpha \boldsymbol{p}_x \cdot \boldsymbol{p}_y - \beta \boldsymbol{p}_x \cdot \boldsymbol{y}.$$

Stability of the primary constraints gives

$$\dot{p}_\alpha = \{p_\alpha, H\} = \boldsymbol{p}_x \cdot \boldsymbol{p}_y =: \psi_1 \qquad \dot{p}_\beta = \{p_\beta, H\} = \boldsymbol{p}_x \cdot \boldsymbol{y} =: \psi_2$$

where ψ_1 and ψ_2 are the secondary constraints. Observe that

$$\{\psi_1, \psi_2\} = -\boldsymbol{p}_x^2.$$

Stability of the secondary constraints gives

$$\dot{\psi}_1 = \{\psi_1, H\} = \beta\, \boldsymbol{p}_x^2 \qquad \dot{\psi}_2 = \{\psi_2, H\} = -\alpha\, \boldsymbol{p}_x^2$$

which lead to the tertiary constraint

$$\psi_3 := \boldsymbol{p}_x^2.$$

The algorithm ends at this stage. Due to the constraint ψ_3, all five constraints of the model are finally first class.

Notice that the rank of the matrix of Poisson brackets among the constraints is not constant. Therefore the existence theorem for Hamiltonian gauge transformations of [13] cannot be applied. In fact we are going to show that no Hamiltonian gauge generators exist in this case. Therefore, the hypothesis in [13] cannot be removed from the assumptions of that existence theorem.

Let us look for a generator of the form (2.13). The more general choice for G_0 is

$$G_0 = f p_\alpha + g p_\beta$$

with f and g functions to be determined with the only restriction that they cannot be constraints (otherwise, G_{H} would be ineffective, $G_{\mathrm{H}} \underset{M_f}{\cong} 0$). Then (2.14*b*) for $k = 1$ leads to

$$G_1 = -f\psi_1 - g\psi_2 + f' p_\alpha + g' p_\beta$$

with f' and g' functions to be determined. Now condition (2.14*c*) imposes

$$\frac{\partial f}{\partial \alpha} = \text{constraint} \qquad \frac{\partial f}{\partial \beta} = \text{constraint}$$
$$\frac{\partial g}{\partial \alpha} = \text{constraint} \qquad \frac{\partial g}{\partial \beta} = \text{constraint}.$$

Next we can construct

$$\begin{aligned} G_2 &= -\{G_1, H\} + \text{PFC} \\ &\cong (f\beta - g\alpha)\psi_3 - (\alpha\{f,\psi_1\} + \beta\{f,\psi_2\} + f')\psi_1 \\ &\quad - (\alpha\{g,\psi_1\} + \beta\{g,\psi_2\} + g')\psi_2 + f''p_\alpha + g''p_\beta \end{aligned}$$

and condition (2.14*c*) leads to

$$g\psi_3 \underset{M_f}{\cong} 0 \qquad f\psi_3 \underset{M_f}{\cong} 0$$

which implies that g and f are constraints. Then the generator $G_{\rm H}$ becomes ineffective and does not generate any transformation of the solutions.

However, this model admits—necessarily non-projectable—Noether gauge transformations, which can be obtained from the results of section 3. Let us look, for instance, for a Noether transformation 'generated' by a combination of all but the primary constraints:

$$G_{\rm H} = f\psi_1 + g\psi_2 + h\psi_3. \tag{4.2}$$

Then we compute, using (2.8)

$$K\cdot G_{\rm H} = (K\cdot f)\,\chi_1 + (K\cdot g)\,\chi_2 + \big(K\cdot h - \mathrm{FL}^*(\alpha g - \beta f)\big)\,\chi_3$$

where $\chi_i = \mathrm{FL}^*(\psi_i)$ are the Lagrangian constraints. Therefore $K\cdot G_{\rm H} \underset{V_1}{\simeq} 0$ is equivalent to

$$K\cdot h = \mathrm{FL}^*(\alpha g - \beta f).$$

It is just a matter of using a function h such that $K\cdot h$ is projectable. In this case, this means that $K\cdot h$ does not depend on $\dot\alpha$ nor $\dot\beta$; in other words, h is independent of α and β. Then f and g are in a simple relation, and a Noether transformation (3.6) is obtained with $r^1 = -K\cdot f$ and $r^2 = -K\cdot g$. More particularly, two independent gauge transformations can be obtained from the following 'generators', which contain the arbitrary functions of time ϵ_1 and ϵ_2:

(i) $G_{\rm H} = -(\dot\epsilon_1(t)/\beta)\psi_1 + \epsilon_1(t)\psi_3$. The corresponding Noether transformation is

$$\delta x = \frac{\dot\epsilon_1}{\alpha}\left(y + \frac{\dot x}{\beta}\right) - \frac{2\epsilon_1}{\alpha}\dot y \qquad \delta\alpha = (\dot\epsilon_1/\beta)^{\cdot}$$
$$\delta y = \frac{\dot\epsilon_1}{\alpha\beta}\dot y \qquad \delta\beta = 0.$$

(ii) $G_{\rm H} = (\dot\epsilon_2(t)/\alpha)\psi_2 + \epsilon_2(t)\psi_3$. Now the transformation is

$$\delta x = \frac{\dot\epsilon_2}{\alpha}y - \frac{2\epsilon_2}{\alpha}\dot y \qquad \delta\alpha = 0$$
$$\delta y = 0 \qquad \delta\beta = -(\dot\epsilon_2/\alpha)^{\cdot}\,.$$

Both transformations fulfil (3.2) with the corresponding $G = FL^*(G_{\rm H})$.

Notice that tha simplest form of a gauge transformation is obtained by chosing $h = 0$, $f = -\alpha\epsilon(t)$ and $g = -\beta\epsilon(t)$. Then

$$G_{\rm H} = \epsilon(t)H = -\epsilon(\alpha\psi_1 + \beta\psi_2)$$

which satisfies

$$K\cdot G_{\rm H} + (\dot{\epsilon\alpha})\,\chi_1 + (\dot{\epsilon\beta})\,\chi_2 = 0.$$

The corresponding Noether transformation is

$$\delta x = \epsilon\dot x \qquad \delta\alpha = (\dot{\epsilon\alpha}) \qquad \delta y = \epsilon\dot y \qquad \delta\beta = (\dot{\epsilon\beta}).$$

Indeed, it can be proven in general that if the Hamiltonian H is a secondary constraint then $G_{\rm H} = \epsilon(t)H$ 'generates in the sense of (3.6)' a Noether transformation.

We have shown therefore that this model exhibits non-trivial Lagrangian Noether gauge transformations, whereas no Hamiltonian gauge generators exist.

5. Example: the bosonic string

The Lagrangian density of the bosonic string is [25, 26]

$$\mathcal{L} = \frac{\sqrt{-g}}{2}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu$$

where $g = g_{00}g_{11} - g_{01}^2$. Using its Hamiltonian constraints and the algorithm described in section 2, one can find a Hamiltonian gauge generator $G_{\rm H}$ which yields its canonical gauge transformations [23]. But these tranformations, when read in Lagrangian formalism, do not have the well known covariant form

$$\delta x^\mu = \epsilon^\alpha\partial_\alpha x^\mu \tag{5.1a}$$

$$\delta g_{\alpha\beta} = \Lambda g_{\alpha\beta} + \epsilon^\gamma\partial_\gamma g_{\alpha\beta} + \partial_\alpha\epsilon^\gamma g_{\gamma\beta} + \partial_\beta\epsilon^\gamma g_{\alpha\gamma} \tag{5.1b}$$

which is not projectable. This transformation can be obtained from the Hamiltonian transformation through a change of gauge parameters [23].

The procedure we have introduced in this paper can be applied to obtain directly the covariant gauge transformations from the Hamiltonian constraints, as we are going to sketch.

The Hamiltonian constraints are: the primary constraints, the momenta $\pi^{\alpha\beta}$ of $g_{\alpha\beta}$, which are equivalent to

$$\phi_W = g_{00}\pi^{00} + g_{01}\pi^{01} + g_{11}\pi^{11}$$
$$\phi_H = \frac{2\sqrt{-g}}{g_{00}}(g_{01}\pi^{01} + g_{11}\pi^{11})$$
$$\phi_T = \frac{1}{g_{00}}\left((2g_{01}^2 - g_{00}g_{11})\pi^{01} + 2g_{01}g_{11}\pi^{11}\right)$$

and the secondary constraints, which can be written as

$$H = \tfrac{1}{2}(p^2 + x'^2) \qquad T = p \cdot x'.$$

All them are related by $\dot{\phi}_W = 0$, $\dot{\phi}_H = H$, and $\dot{\phi}_T = T$. The Hamiltonian density is also a secondary constraint

$$\mathcal{H} = -\frac{\sqrt{-g}}{g_{11}} H + \frac{g_{01}}{g_{11}} T.$$

From the primary Hamiltonian constraints we obtain the kernel $\gamma_{\alpha\beta}(\sigma, \sigma')$ of $W(\sigma, \sigma')$, which is given by the $\Gamma_{\alpha\beta}(\sigma) = \partial / \partial \dot{g}_{\alpha\beta}$; equivalently we can use

$$\begin{aligned}
\Gamma_W &= \left(g_{00} \frac{\partial}{\partial \dot{g}_{00}} + g_{01} \frac{\partial}{\partial \dot{g}_{01}} + g_{11} \frac{\partial}{\partial \dot{g}_{11}} \right) \\
\Gamma_H &= \frac{2\sqrt{-g}}{g_{00}} \left(g_{01} \frac{\partial}{\partial \dot{g}_{01}} + g_{11} \frac{\partial}{\partial \dot{g}_{11}} \right) \\
\Gamma_T &= \frac{1}{g_{00}} \left((2g_{01}^2 - g_{00} g_{11}) \frac{\partial}{\partial \dot{g}_{01}} + 2 g_{01} g_{11} \frac{\partial}{\partial \dot{g}_{11}} \right).
\end{aligned}$$

Finally, the primary Lagrangian constraints are the $K \cdot \pi^{\alpha\beta}(\sigma)$, which are not independent. We will use instead the following constraints: $K \cdot \phi_H(\sigma) = \mathrm{FL}^*(H(\sigma))$ and $K \cdot \phi_T(\sigma) = \mathrm{FL}^*(T(\sigma))$, whereas $K \cdot \phi_W$ is identically zero.

As in the previous example, we can look for a 'generator' G_{H} made up of secondary constraints with arbitrary parameters. Let us consider the most general G_{H}, which can be taken as $G_{\mathrm{H}} = \int \mathrm{d}\sigma \, (\epsilon^0 \mathcal{H} + \epsilon^1 T)$. It satisfies $K \cdot G_{\mathrm{H}} = -\int \mathrm{d}\sigma \, (r_H \mathrm{FL}^*(H) + r_T \mathrm{FL}^*(T)) \underset{V_1}{\simeq} 0$, which determines

$$\begin{aligned}
r_H(\sigma) &= \left(\epsilon^0 \frac{\sqrt{-g}}{g_{11}} \right)^{\cdot} - 2\epsilon^{0\prime} \frac{\sqrt{-g}\, g_{01}}{g_{11}^2} + \epsilon^1 \left(\frac{\sqrt{-g}}{g_{11}} \right)' - \epsilon^{1\prime} \frac{\sqrt{-g}}{g_{11}} \\
r_T(\sigma) &= -\left(\epsilon^0 \frac{g_{01}}{g_{11}} \right)^{\cdot} + \epsilon^{0\prime} \frac{g_{01}^2 - g}{g_{11}^2} + -\dot{\epsilon}^1 - \epsilon^1 \left(\frac{g_{01}}{g_{11}} \right)' + \epsilon^{1\prime} \frac{g_{01}}{g_{11}}.
\end{aligned}$$

The function r_W corresponding to the vanishing Lagrangian constraint $K \cdot \phi_W$ remains arbitrary, and we can take advantage of this to introduce a new gauge parameter (arbitrary function) Λ.

Once r_W is fixed, we use (3.6) to determine the gauge transformation. It turns out that its most simple form is achieved when we take

$$r_W = \Lambda + \frac{1}{g_{00}} \left(\epsilon^0 \dot{g}_{00} + 2\dot{\epsilon}^0 g_{00} + \epsilon^1 g_{00}' + 2\dot{\epsilon}^1 g_{01} \right)$$

which yields (5.1).

6. Conclusions

In this paper we have obtained a neat characterization for Noether transformations (of Lagrangian formalism): they are 'generated' by a function $G_{\rm H}(q, p; t)$ (of Hamiltonian formalism) which satisfies $K \cdot G_{\rm H} \underset{V_1}{\simeq} 0$. This result is completely general: it includes both the cases of gauge and rigid transformations as well as the projectable and the non-projectable transformations. Owing to the dynamic contents of the operator K, $G_{\rm H}$ is a constant of motion. The case of gauge transformations corresponds to a $G_{\rm H}$ made up by arbitrary functions of time and Hamiltonian constraints.

The case of non-projectable Noether transformations is especially relevant because they cannot have an associated transformation in phase space. The example in section 4 shows that there are cases for which no Hamiltonian gauge generators exist, even though the dynamics contains arbitrary functions. This fact is compatible with the existence of Noether gauge transformations in velocity space. The same example proves that the hypothesis of constant rank of the Poisson bracket among constraints cannot be removed from the theorem of existence of a complete set of independent gauge transformations in phase space [13]. On the other hand, as shown in the second example, our procedure allows us to obtain the covariant gauge transformations even when this is not possible within the Hamiltonian framework.

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References

[1] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University)

[2] Batlle C, Gomis J, Pons J M and Román-Roy N 1986 Equivalence between the Lagrangian and Hamiltonian formalisms for constrained systems *J. Math. Phys.* **27** 2953–62

[3] Gràcia X and Pons J M 1988 Gauge transformations, Dirac's conjecture and degrees of freedom for constrained systems *Ann. Phys.* **187** 355–68

[4] Gotay M J and Nester J M 1979 Presymplectic Lagrangian systems I: the constraint algorithm and the equaivalence theorem *Ann. Inst. H. Poincaré* **30** 129–42

[5] Pons J M 1988 New relations between Hamiltonian and Lagrangian constraints *J. Phys. A: Math. Gen.* **21** 2705–15

[6] Gràcia X and Pons J M 1989 On an evolution operator connecting Lagrangian and Hamiltonian formalisms *Lett. Math. Phys.* **17** 175–80

[7] Anderson J L and Bergmann P G 1951 Constraints in covariant field theory *Phys. Rev.* **83** 1018–25

[8] Castellani L 1982 Symmetries in constrained Hamiltonian dynamics *Ann. Phys.* **143** 357–71

[9] Sugano R, Saito Y and Kimura T 1986 Generator of gauge transformations in phase space and in velocity phase space *Prog. Theor. Phys.* **76** 283–301

[10] Henneaux M, Teitelboim C and Zanelli J 1990 Gauge invariance and degree of freedom count *Nucl. Phys.* B **332** 167–88

[11] DeWitt B S 1964 Dynamical theory of groups and fields *Relativity, Groups and Topology* (New York: Gordon and Breach)

[12] Batalin I A and Vilkovisky G A 1985 Existence theorem for gauge algebra *J. Math. Phys.* **26** 172–84

[13] Gomis J, Henneaux M and Pons J M 1990 Existence theorem for gauge symmetries in Hamiltonian constrained systems *Class. Quantum Grav.* 7 1089–96

[14] Noether A E 1918 Invariante variations problems *Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl.* pp 235–57

[15] Bessel-Hagen E 1921 Uber die Erhaltungssätze der Elektrodynamik *Math. Ann.* **84** 258–76

[16] Hill E L 1951 Hamilton's principle and the conservation theorems of mathematical physics *Rev. Mod. Phys.* **23** 253–60

[17] Rosen J 1974 Generalized Noether's theorem *Ann. Phys.* **82** 54–69

[18] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)

[19] Lusanna L 1991 The second Noether theorem as the basis of the theory of singular Lagrangians and Hamiltonian constraints *Riv. Nuovo Cim.* **14** 3

[20] Cariñena J F, López C and Martínez E 1989 A new approach to the converse of Noether's theorem *J. Phys. A: Math. Gen.* **22** 4777–86

[21] Ferrario C and Passerini A 1990 Symmetries and constants of motion for constrained Lagrangian systems: a presymplectic version of the Noether theorem *J. Phys. A: Math. Gen.* **23** 5061–81

[22] Kamimura K 1982 Singular Lagrangian and constrained Hamiltonian systems, generalized canonical formalism *Nuovo Cim.* B **68** 33–54

[23] Batlle C, Gomis J, Gràcia X and Pons J M 1989 Noether's theorem and gauge transformations: Application to the bosonic string and CP_2^{n-1} model *J. Math. Phys.* **30** 1345–50

[24] Gomis J, Kamimura K and Pons J M 1986 To construct gauge transformations from singular Lagrangians *Europhys. Lett.* 2 187–94

[25] Brink L, DiVecchia P and Howe P 1976 A locally supersymmetric and reparametrization invariant action for the spinning string *Phys. Lett.* B **65** 471–4

[26] Batlle C, Gomis J and Pons J M 1986 Hamiltonian and Lagrangian constraints of the bosonic string *Phys. Rev.* D **34** 2430–2